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USE OF HYDRAULIC RESONANCE IN A PIPELINE WITH A GAS CAVITY TO CREATE A NONSTATIONARY JET

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High-velocity jets of liquid are widely employed in engineering for breaking down and cutting different materials. In some cases a steady jet is employed, but the results of [1] show that it is better to employ a nonstationary jet, because in this case the main mechanism of erosion of material is the high pressure of the hydraulic impact of the jet. In [2] several aspects of the creation of a nonstationary jet, emanating from a nozzle at the end of a pipe, are studied in application to hydraulic extraction of useful minerals. The oscillatory process in a pipe entirely filled with liquid is studied by the method of mathematical modeling. The nonstationary state of the jet was created either by pulsating the flow rate of the liquid at the pipe inlet or by periodically changing the cross section of the jet with the help of an oscillating valve.

It is well known that the presence of an air cavity in a liquid-filled pipe could give rise to significant oscillations of the velocity and pressure of the liquid in different nonstationary processes [3-7]. This is explained by the appearance of characteristic oscillations of a column of liquid with a frequency which is determined simultaneously by the parameters of the cavity, the liquid, and the pipe [8, 9]. Pressure oscillations during transient processes in a pipe are usually regarded as an undesirable phenomenon, so that the parameters of the air chamber are chosen so as to dampen these oscillations. At the same time there exist hydroimpact systems in which the oscillations of the liquid in a pipe without a gas cavity are specially created with the help of a valve which is periodically covered in order to obtain pressure pulses [10]. Since the presence of a gas cavity in a pipe containing liquid can lead to a significant increase in pressure [6, 7] this effect could be useful in obtaining high pressures, as pointed out in [7].

Physical-Mathematical Formulation of the Problem. In this paper we shall study the possibility of employing resonance oscillations of a liquid in a pipe with a gas cavity and a nozzle at the end (Fig. 1) to create a high-velocity pulsating jet. It is assumed that the oscillations arise as a result of modulation of the pressure $\tilde{p}_{in}(\tilde{t})$, which varies according to the law $\tilde{p}_{in}(\tilde{t}) = \tilde{p}_0 + \Delta\tilde{p}\cos(2\pi\tilde{t}/\tilde{T})$, at the pipe inlet. Here $\tilde{p}_0$ is the stationary pressure in the system, and $\Delta\tilde{p}$ and $\tilde{T}$ are the amplitude and period of the pulsation arising during pump operation. We shall study the problem in the approximation of an incompressible liquid, making the assumption that the velocity of the liquid is identical in all sections of the pipe. Neglecting the propagation time of the disturbances along the pipe in this manner will be justified if $\tilde{c}\tilde{T} \gg \tilde{L}$ ($\tilde{L}$ and $\tilde{c}$ are the length of the pipe and the velocity of propagation of the wave). We shall write the equations of motion of the liquid, the change in the volume of the gas cavity, and the adiabatic compression of the gas in the form

$$\tilde{\rho}\tilde{L}d\tilde{u}/d\tilde{t} = \tilde{p}_{in} - \tilde{p} - \lambda\tilde{L}\tilde{\rho}|\tilde{u}|\tilde{u}/2\tilde{D},$$
$$d\tilde{V}/d\tilde{t} = -\tilde{f}_0\tilde{u} + \tilde{s}_0\sqrt{2(\tilde{p}-\tilde{p}_a)/\tilde{\rho}}, \quad \tilde{p}\tilde{V}^\gamma = \tilde{p}_0\tilde{V}_0^\gamma, \tag{1}$$

where $\tilde{u}$, $\tilde{\rho}$, and $\tilde{p}$ are the velocity, density, and pressure of the liquid at the end of the pipe; $\tilde{f}_0$ and $\tilde{s}_0$ are the cross-sectional area of the pipe and the effective area of the nozzle; $\sqrt{2(\tilde{p}-\tilde{p}_a)/\tilde{\rho}}$ is the velocity of the jet; $\tilde{p}_a$ is the atmospheric pressure; λ is the coefficient of friction at the wall; $\tilde{D}$ is the diameter of the pipe; and, γ is the adiabatic index.

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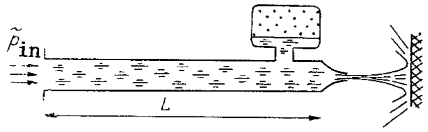

Fig. 1

We shall determine the condition under which the pressure losses due to friction against the pipe wall can be neglected. Equating in order of magnitude the first two terms in the second equation in (1), we obtain the estimate $\tilde{u} \sim \tilde{V}_0/\tilde{f}_0\tilde{T}$ for the characteristic velocity. In order to neglect the friction as compared with the inertial term in the equation of motion it is necessary that $\lambda\tilde{u}\tilde{T}/\tilde{D} \ll 1$; from here we obtain the condition $\lambda\tilde{V}_0/\tilde{D}\tilde{f}_0 \ll 1$. Since the characteristic value $\lambda \sim 10^{-2}$, for $\tilde{V}_0/\tilde{f}_0\tilde{L} \sim 1$ this condition holds for not very long pipes ($\tilde{L}/\tilde{D} \sim 10$).

Neglecting friction in the first of the equations (1) and transforming to dimensionless variables $p = \tilde{p}/\tilde{p}_0$, $u = \tilde{u}/\sqrt{\tilde{p}_0\tilde{V}_0/\tilde{\rho}\tilde{L}\gamma\tilde{f}_0}$, $t = \tilde{t}/\sqrt{\tilde{\rho}\tilde{L}\tilde{V}_0/\tilde{p}_0\gamma\tilde{f}_0}$ we obtain, eliminating $\tilde{V}$ from (1),

$$du/dt = 1 + q\cos\omega t - p, \quad dp/dt = p^{(1+\gamma)/\gamma}\,(u - 2\varepsilon\sqrt{p - p_a}). \tag{2}$$

Here $q = \Delta\tilde{p}/\tilde{p}_0$; $p_a = \tilde{p}_a/\tilde{p}_0$; $\varepsilon = s_0\sqrt{\gamma\tilde{f}_0\tilde{L}}/\tilde{f}_0\sqrt{2\tilde{V}_0}$; and $\omega = 2\pi\sqrt{\tilde{\rho}\tilde{L}\tilde{V}_0}/\tilde{T}\sqrt{\tilde{p}_0\gamma\tilde{f}_0}$. We shall seek the steady-state periodic solution of (2), corresponding to forced oscillations with fixed values of q and ω (amplitude and frequency of pressure pulsations at the inlet). Thus we arrive at the formulation of the problem of nonlinear oscillations. The solution of the formulated problem can be constructed in the general case, for example, by integrating the system (2) numerically. We shall first study the particular case of small oscillations.

Approximate Solution in the Linear Case. In the absence of pressure pulsations at the pipe inlet (q = 0) the stationary values are p = 1 and $u = 2\varepsilon\sqrt{1 - p_a}$. We linearize (2), setting $p = 1 + \bar{p}$, $u = 2\varepsilon\sqrt{1 - p_a} + \bar{u}$, $\bar{p} \ll 1 - p_a$, $q \ll 1 - p_a$. As a result we obtain

$$d\bar{u}/dt = -\bar{p} + q\cos\omega t, \quad d\bar{p}/dt = \bar{u} - 2k\bar{p}, \quad k = \varepsilon/2\sqrt{1 - p_a}.$$

Eliminating $\bar{u}$ we arrive at the well-known equation for oscillations of a damped linear oscillator

$$\ddot{\bar{p}} + 2k\dot{\bar{p}} + \bar{p} = q\cos\omega t. \tag{3}$$

Here the coefficient k, which plays the role of friction, takes into account the energy losses in the system owing to outflow of liquid through the nozzle. The solution (3), corresponding to forced oscillations, is

$$\bar{p} = A\cos(\omega t - \alpha), \quad A = q/\sqrt{(\omega^2 - 1)^2 + 4k^2\omega^2}, \quad \mathrm{tg}\,\alpha = 2k\omega/(1 - \omega^2). \tag{4}$$

The maximum value of the amplitude with $\omega_{res} = \sqrt{1 - 2k^2}$ and the width of the resonance peak are determined by the quantity k. As the effective area of the nozzle is decreased the resonance peak becomes higher and narrower, remaining quite symmetric relative to $\omega = \sqrt{1 - 2k^2}$.

Nonlinear Approximate Analytical Solution. We shall study resonance in the nonlinear formulation, making the assumptions that $q \ll 1 - p_a$ and $\varepsilon \ll 1$. We introduce $w = 1 - p^{-1/\gamma}$; then, eliminating u, the system (2) can be reduced to the equation

$$\ddot{w} = \frac{1}{\gamma}\left[1 - (1 - w)^{-\gamma}\right] + \frac{1}{\gamma}q\cos\omega t - \frac{\varepsilon\dot{w}\,(1 - w)^{-(\gamma+2)/2}}{\sqrt{1 - p_a(1 - w)^{\gamma}}}.$$

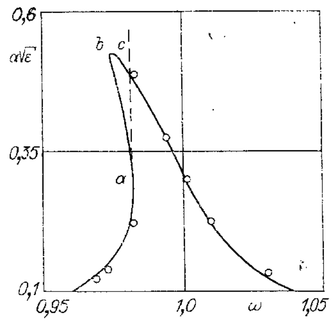

Fig. 2

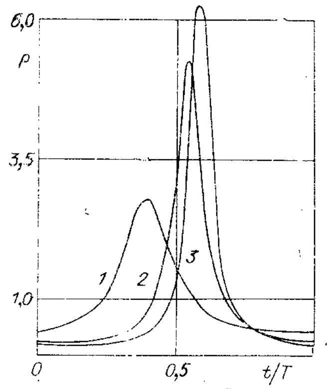

Fig. 3.

For small values of w, expanding $(1-w)^{-\gamma}$ in a Taylor series around w = 0, we obtain with accuracy to w^3

$$\ddot{w}+w+\alpha_2 w^2+\alpha_3 w^3+\varepsilon\dot{w}(1-w)^{-(\gamma+2)/2}/$$
$$/\sqrt{1-p_a+\gamma p_a(w-\alpha_2 w^2+\alpha_3 w^3)}=(q/\gamma)\cos\omega t,$$
$$\alpha_2=(\gamma+1)/2,\ \alpha_3=(\gamma+1)(\gamma+2)/6. \tag{5}$$

We renormalize (5), making the substitution $w=\sqrt{\varepsilon^*}y$, where $\varepsilon^*=\varepsilon/\sqrt{1-p_a}$. Retaining on the left side only terms of order $\sqrt{\varepsilon^*}$, ε^*, we obtain

$$\ddot{y}+y+\sqrt{\varepsilon^*}\alpha_2 y^2+\varepsilon^*(\alpha_3 y^3+\dot{y})=\varepsilon^* f\cos\omega t,\ f=q/\gamma\varepsilon^*\sqrt{\varepsilon^*}. \tag{6}$$

We seek the approximate solution of (6) by the method of many scales [11], using the variables $T_0=t$, $T_1=\sqrt{\varepsilon^*}t$, $T_2=\varepsilon^* t$ and representing y(t) in the form

$$y=y_0(T_0,T_1,T_2)+\sqrt{\varepsilon^*}y_1(T_0,T_1,T_2)+\varepsilon^* y_2(T_0,T_1,T_2). \tag{7}$$

Substituting (7) into (6) and equating the coefficients of like powers of ε* on both sides of the equation we obtain

$$D_0^2 y_0+y_0=0; \tag{8}$$
$$D_0^2 y_1+y_1=-2D_1D_0y_0-\alpha_2 y_0^2; \tag{9}$$
$$D_0^2 y_2+y_2=-2D_1D_0y_1-2\alpha_2 y_0 y_1-2D_2D_0y_0-$$
$$-D_1^2y_0-D_0y_0-\alpha_3 y_0^3+f\cos\omega T_0. \tag{10}$$

Here $D_j=\partial/\partial T_j\ (j=0,1,2)$. We write the solution (8) in the form $y_0=A(T_1,T_2)e^{iT_0}+\bar{A}(T_1,T_2)e^{-iT_0}$ (A and $\bar{A}$ are complex conjugate quantities). From here we obtain for (9)

$$D_0^2y_1+y_1=-\left(2i\,\partial A/\partial T_1\cdot e^{iT_0}+\alpha_2A^2e^{2iT_0}\right)-(\text{c.c.})-2\alpha_2A\bar{A},$$

where c.c. denotes the complex-conjugate term. Then the requirement that there be no secular terms in the solution of this equation leads to the condition

$$\partial A/\partial T_1=0. \tag{11}$$

The solution of the inhomogeneous equation (9) assumes the form

$$y_1=(\alpha_2A^2/3)e^{2iT_0}+(\text{c.c.})-2\alpha_2A\bar{A}.$$

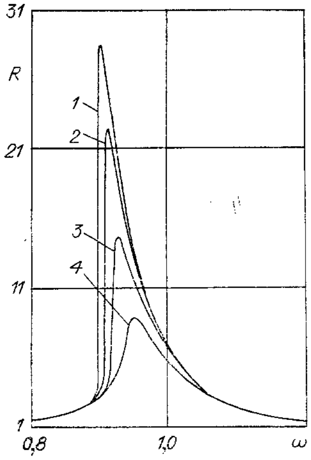

Fig. 4

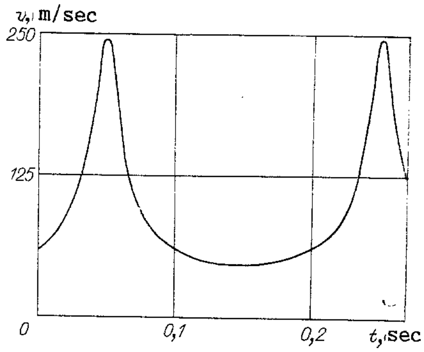

Fig. 5

Using the expressions obtained for y_0 and y_1 and the condition (11) we write (10) as follows:

$$D_0^2 y_2 + y_2 = (\mu_1 A^2 \bar{A} - 2i\,\partial A/\partial T_2 - iA)\,e^{iT_0} - \mu_2 A^3 e^{3iT_0} + \\ + f e^{i\omega T_0}/2 + (\text{c.c.}) \quad \mu_1 = (\gamma + 1)(2\gamma - 1)/6, \quad \mu_2 = (\gamma + 1)(2\gamma + 3)/6. \tag{12}$$

Introducing in the well-known manner the detuning parameter $\sigma = (\omega - 1)/\varepsilon^*$ and writing $e^{i\omega T_0} = e^{iT_0} e^{i\sigma T_2}$, we eliminate the secular terms in the solution (12), assuming that the following equation is valid:

$$\mu_1 A^2 \bar{A} - 2i\,\partial A/\partial T_2 - iA + f e^{i\sigma T_2}/2 = 0. \tag{13}$$

Representing A in the exponential form $A = ae^{i\beta}/2$, where a and β are real quantities, and separating real and imaginary parts in (13) we arrive at a system of two equations for a and β. We transform this system into an autonomous system, introducing the variable $\varphi = \sigma T_2 - \beta$:

$$\partial a/\partial T_2 = -a/2 + (f/2)\sin\varphi,$$
$$a\partial\varphi/\partial T_2 = a\sigma + \mu_1 a^3/8 + (f/2)\cos\varphi.$$

From here we find the steady-state values of a and φ, which satisfy the relations $a = f\sin\varphi$, $2a(\sigma + \mu_1 a^2/8) = -f\cos\varphi$, and the amplitude-frequency characteristic (AFC) of the system

$$a^2 + 4a^2(\sigma + \mu_1 a^2/8)^2 = f^2. \tag{14}$$

We note that the AFC obtained is close to that of the Duffing equation [11], but it differs from the latter by the sign of the term $\mu_1 a^2/8$. The jump phenomenon, i.e., a jump in the amplitude owing to the fact that a(ω) is a three-valued function for some ω > 1, is well known for the AFC of Duffing's equation. This phenomenon also occurs for the oscillatory system under study.

We shall study the condition under which a jump in the amplitude is possible for the AFC (14). Solving the equation for ω we find

$$\omega - 1 = -\mu_1 a^2 \varepsilon^*/8 \pm (1/2)\varepsilon^* \sqrt{(f/a)^2 - 1}. \tag{15}$$

The function a(ω) will be multivalued, if dω/da = 0 for some a. The values of a satisfying this condition can lie only on the lower branch of (15) (the minus sign in front of the square root). From here $d\omega/da = \varepsilon^* f^2/2a^3\sqrt{(f/a)^2 - 1} - \mu_1 a\varepsilon^*/4 = 0$. We shall find the values of ε* and f for which this equation has a solution for a; for this we write the equation in the form

$$\psi(z) = \mu_1\varepsilon^*/2, \tag{16}$$

where $z = (f\varepsilon^*/a)^2$; and $\psi(z) = z^2/f^2\varepsilon^{*2}\sqrt{z-\varepsilon^{*2}}$. The function $\psi(z)$ has a minimum at $z_{min} = 4\varepsilon^{*2}/3$, $\psi_{min} = 16\sqrt{3}\varepsilon^*/9f^2$. Therefore Eq. (16) has two solutions if $\psi_{min} < \mu_1\varepsilon^*/2$. Expressing f in terms of q and ε^* we obtain the relation

$$\varepsilon^{*3}/q^2 < 9\mu_1/(32\sqrt{3}\gamma^2). \tag{17}$$

Thus for a given value of q a jump in the amplitude can occur only if the area of the nozzle is sufficiently small (small values of ε^*). We note that for the AFC (14), unlike Duffing's equation, the region where the function $a(\omega)$ is not single-valued lies to the left of $\omega = 1$. In the nonlinear approximation under study the resonance peak of the AFC (14) is asymmetric, since on the whole it is inclined toward lower frequencies.

We shall write the solutions obtained y_0 and y_1 in the real form

$$y_0 = (1/2)\left(e^{i(T_0+\beta)} + e^{-i(T_0+\beta)}\right) = a\cos(\omega t - \varphi),$$
$$y_1 = (\alpha_2 a^2/2)[(1/3)\cos(2\omega t - 2\varphi) - 1].$$

From here we find the approximate solution (5), returning to the variable w and using two terms of the expansion (7):

$$w = a\sqrt{\varepsilon^*}\cos(\omega t - \varphi) + (a^2\varepsilon^*\alpha_2/2)[(1/3)\cos(2\omega t - 2\varphi) - 1]. \tag{18}$$

For $a \ll 1$ the second term in (18) can be neglected. Then $p = (1-w)^{-\gamma} \approx 1 + \gamma a\sqrt{\varepsilon^*}\cos(\omega t - \varphi)$. It is not difficult to show that near resonance $\omega \approx 1$ this solution transforms into the solution (4) of the linearized equation (3). Indeed, if $a^2 \ll \sigma$, then we obtain from (14) $a = f/\sqrt{1+4\sigma^2}$, whence the amplitude of the pressure $\gamma a\sqrt{\varepsilon^*} = q/\varepsilon^*\sqrt{1+4(\omega-1)^2/\varepsilon^{*2}} \approx q/\sqrt{\varepsilon^{*2}+(\omega-1)^2(\omega+1)^2} = q/\sqrt{4k^2+(\omega^2-1)^2}$ $(\omega \approx 1,\ \omega+1 \approx 2)$.

The nonlinear analytic solution found in the small amplitude approximation makes it possible to study the qualitative properties of the process under study.

Numerical Solution in the General Case of Large Amplitudes. Discussion of Results. To study steady forced oscillations in the general case the system (2) was integrated by a rigidly stable numerical method [12] with fifth-order accuracy and the initial conditions $p(0) = 1$, $u(0) = 2\varepsilon\sqrt{1-p_a}$ for fixed values of q and ω until a periodic regime was established. The calculations were performed for $\rho = 10^3$ kg/m^3 (water) and $\gamma = 1.4$ (air). Figure 2 shows curves of $a\sqrt{\varepsilon^*}$ versus the frequency, calculated from (14) for $q = 10^{-2}$, $\varepsilon^* = 1.34\cdot10^{-2}$, satisfying (17). The circles show the results of integration of the system (2). One can see that for small values of q the numerical solution is in good agreement with the approximate analytical solution. For frequencies $0.974 < \omega < 0.982$, $a(\omega)$ is a three-valued function and the lower branch is realized in the numerical calculation. The sections ab and bc of the curve are unstable. As the frequency is varied a jump-like change in amplitude occurs at the point $\omega = 0.982$ (appearance of a jump).

For $q \gg 10^{-2}$ the form of the steady oscillations p(t) is strongly nonsinusoidal owing to the nonlinearity (2). Figure 3 shows the time dependence of the pressure during the period with $p_a = 0.1$, $\tilde{s}_0/\tilde{f}_0 = 0.035$, $\tilde{V}_0/\tilde{f}_0\tilde{L} = 0.1$, $\omega = 0.958$, for different values of q (q = 0.1 (1), 0.3 (2), and 0.6 (3)). One can see that the pressure oscillations at the end of the pipe are much greater than the pulsations at the inlet. As q is increased not only the maximum value of p_{max} increases, but the steepness of the peak increases also. The maximum value and rate of increase of the velocity of the jet $\tilde{v}(\tilde{t}) = \sqrt{2(\tilde{p}-\tilde{p}_a)/\tilde{\rho}}$, which determine the effectiveness of its hydroimpact action on the material being broken down [2], also increase. We shall characterize the double amplitude of the pressure oscillations at the nozzle by the quantity $R = p_{max}/p_{min}$. The curves $R(\omega)$, replacing in the case of large amplitude the AFC (14), are presented in Fig. 4 for q = 0.1 and p_a = 0.02 for different values of $\tilde{s}_0/\tilde{f}_0$ ($\tilde{s}_0/\tilde{f}_0$ = 0.035 (1), 0.05 (2), 0.075 (3), and 0.1 (4)). The form of the characteristics R(ω) for large values of q qualitatively agrees with the AFC (14) for small amplitudes. The curves R(ω) also slope toward lower frequencies, the left slope being steeper, and they have a maximum for ω < 1. As the area of the nozzle is decreased the resonance peaks become steeper and higher, and the maximum of the curve shifts to the left. For small values of $\tilde{s}_0/\tilde{f}_0$ a jump is observed for some ω < 1, i.e., R(ω) becomes discontinuous, which also is in qualitative agreement with the condition (17), obtained for q << 1. We note the significant maximum values R_{max} ~ 10 for low degree of modulation of the pressure at the inlet q = 0.1. This indicates that the resonance regime is highly effective for obtaining significant pressure oscillations in front of the nozzle and therefore a high velocity of the outflowing jet.

The steady oscillatory regime of the fluid flow, as follows from (2), is determined completely by the parameters q, ω, p_a, and ε*. For a pipe with concrete values of $\tilde{L}$, $\tilde{f}_0$, and $\tilde{s}_0$ with fixed values of $\tilde{p}_0$, $\Delta\tilde{p}$, and $\tilde{T}$ the condition of a resonance maximum can be realized by choosing the volume of the gas cavity $\tilde{V}_0$, knowing the dimensionless frequency ω_{res}. In the case of small amplitude ω_{res} must give a maximum of the AFC (14) and is determined by the condition da/dω = 0. As one can see from (15) the equivalent condition $d\omega/da \to \infty$ holds for a = f. From here $\omega_{res} = 1 - \mu_1 q^2/8\gamma^2\varepsilon^{*2}$. On the other hand, by definition, $\omega\varepsilon^* = 2\pi\tilde{s}_0\tilde{L}/\tilde{f}_0\tilde{T}\sqrt{2(\tilde{p}_0 - \tilde{p}_a)/\tilde{\rho}} \equiv B$ is known. We have for ε* the quadratic equation $\varepsilon^{*2} - B\varepsilon^* - \varepsilon_q^2 = 0$ $(\varepsilon_q^2 = \mu_1 q^2/8\gamma^2)$. From here we find $\varepsilon^* = B/2 + \sqrt{\varepsilon_q^2 + B^2/4}$, $\omega_{res} = B\Big/\left(B/2 + \sqrt{\varepsilon_q^2 + B^2/4}\right)$, $\tilde{V}_0/\tilde{f}_0\tilde{L} = (\omega_{res}/2\pi)^2\gamma\tilde{p}_0\tilde{T}^2/\tilde{\rho}\tilde{L}^2$. In the general case of large amplitudes it is necessary to construct numerically the dependence R(ω) for different values of ε*, and then determine from them the resonance frequencies and find the intersection of the graphs $\omega_{res} = \omega_{res}(\varepsilon^*)$ and $\omega_{res} = B/\varepsilon^*$.

Neglecting the compressibility of the liquid, as done above, leads to the condition for the volume of the gas $\tilde{V}_0/\tilde{f}_0\tilde{L} \gg \tilde{p}_0/\tilde{\rho}\tilde{c}^2$. As an example we shall study a pipe of length $\tilde{L}$ = 2.74 m, filled with water at a pressure $\tilde{p}_0$ = 5 MPa with a nozzle $\tilde{s}_0/\tilde{f}_0$ = 0.035. Assume that at the inlet $\Delta\tilde{p}$ = 0.5 MPa and $\tilde{T}$ = 0.272 sec. For $\tilde{V}_0/\tilde{f}_0\tilde{L} = 1$ the dimensionless quantities are $q = 0.1$, $p_a = 0.02$, $\varepsilon^* = 2.93\cdot10^{-2}$, $\omega = 0.9064$, which corresponds to a maximum of the curve 1 in Fig. 4. The time dependence of the velocity of the jet $\tilde{v}(\tilde{t})$ for this example is shown in Fig. 5. The maximum hydroimpact pressure of the jet [2] $\tilde{p}_g = \tilde{\rho}\tilde{c}\,(\tilde{v}_{max} - \tilde{v}_{min}) = 276.7$ MPa is high enough to breakdown many materials.

Thus the process of forced nonlinear oscillations of a liquid in a pipe with a gas cavity which appear when the pressure at the pipe inlet is modulated was studied. An approximate analytical solution was obtained in the small-amplitude approximation. It was shown that the resonance regime can be employed to obtain in the output section of the pipe a pulsating high-velocity jet.

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